JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 943–954 (1996)
ARTICLE NO. 0293

Asymptotic Behavior of Solutions of Impulsive Delay Differential Equations*

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Received May 25, 1995

Effective sufficient conditions for the asymptotic stability of the trivial solution of impulsive delay differential equation

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) = 0 \\ x(t_k^+) - x(t_k) = b_k x(t_k) \end{cases}$$

are obtained by investigating respectively the asymptotic behavior of the nonoscillatory solutions and oscillatory solutions of the equation.

1. INTRODUCTION

The impulsive delay differential equations are adequate mathematical models of numerous processes and phenomena studied in physics, biology, engineering, etc. Despite the great possibilities for applications, the theory of these equations is developing comparatively slowly due to significant difficulties of technical and theoretical character [2, 3]. The theory of impulsive differential equations is considerably richer than the theory of ordinary differential equations and makes a certain development in the recent years [7]. The theory of delay differential equations is also comparatively well developed [4, 5].

The objective of this paper is to investigate the asymptotic properties of solutions of the first order linear impulsive delay differential equations. As

*This work was partially supported by the National Natural Science Foundation of China.

0022-247X/96 $18.00

an application the conditions for asymptotic stability of the zero solution are obtained.

2. PRELIMINARIES

Consider the impulsive delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) = 0, \qquad t \neq t_k \tag{1}$$

$$x(t_k^+) - x(t_k) = b_k x(t_k), \qquad k = 1, 2, \ldots, \tag{2}$$

where $p_i \in C([0,\infty), R)$, $i = 1, 2, \ldots, n$; $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, and b_k, $k = 1, 2, \ldots$ are constants. By developing the method in [5], we obtain sufficient conditions for nonoscillatory solutions and oscillatory solutions of (1) and (2) tending to zero as $t \to \infty$.

Let PC_σ denote the set of functions $\phi\colon [\sigma - \tau_n, \sigma] \to R$ which is continuous in the set $[\sigma - \tau_n, \sigma] \setminus \{t_k\colon k = 1, 2, \ldots\}$ and at the points t_k situated in the interval $(\sigma - \tau_n, \sigma]$ may have discontinuities of the first kind and is continuous from left. For any $\sigma \geq 0$, $\phi \in PC_\sigma$, a function x is said to be a solution of (1) and (2) satisfying the initial value condition

$$x(t) = \phi(t), \qquad t \in [\sigma - \tau_n, \sigma] \tag{3}$$

on the interval $[\sigma, a)$, if $x\colon [\sigma - \tau_n, a) \to R$ satisfies (3) and

(a) for $t \in (\sigma, a)$, $t \neq t_k$, $t \neq t_k + \tau_i$, $x(t)$ is continuously differentiable and satisfies (1);

(b) for $t_k \in [\sigma, a)$, $x(t_k^+)$, $x(t_k^-)$ exist, $x(t_k^-) = x(t_k)$ and satisfies (2);

(c) for $t \in (\sigma, a)$, $t \neq t_k$, $t = t_k + \tau_i$, $x(t)$ is continuous and $x'(t^+)$, $x'(t^-)$ exist.

LEMMA 1. *For any $\sigma \geq 0$ and $\phi \in PC_\sigma$ the initial value problem* (1), (2), *and* (3) *has exactly one solution $x(t)$ on the interval $[\sigma, \infty)$ and*

$$x(t) = y(t) + \sum_{\sigma \leq t_k < t} b_k x(t_k) u_k(t), \tag{4}$$

where $y \in C[\sigma, \infty)$ and $u_k \in C(t_k, \infty)$, $k = 1, 2, \ldots$ are respectively the solutions of the initial value problems

$$y'(t) + \sum_{i=1}^{n} p_i(t)y(t-\tau_i) = 0, \qquad t > \sigma, \tag{5}$$

$$y(t) = \phi(t), \qquad t \in [\sigma - \tau_n, \sigma]. \tag{6}$$

and

$$u_k'(t) + \sum_{i=0}^{n} p_i(t) u_k(t - \tau_i) = 0, \qquad t > t_k, \tag{7}$$

$$u_k(t) = 0, \qquad t \in [t_k - \tau_n, t_k], \qquad u_k(t_k^+) = 1, \tag{8}$$
$$k = 1, 2, \dots .$$

Proof. Let $l = \min\{k : t_k \geq \sigma\}$. Then for $t \in [\sigma, t_l]$ the initial value problem (1), (2), and (3) coincides with the initial value problem (5) and (6), hence $x(t) = y(t)$.

If $t \in (t_l, t_{l+1}]$, denote $z(t) = x(t) - y(t)$, then $z(t)$ satisfies

$$\begin{cases} z'(t) + \sum_{i=1}^{n} p_i(t) z(t - \tau_i) = 0, & t \in (t_l, t_{l+1}], \\ z(t) = 0, & t \in [t_l - \tau_n, t_l], \\ z(t_l^+) = b_l x(t_l), \end{cases}$$

so $z(t) = b_l x(t_l) u_l(t)$ and the formula (4) holds for $t \in (t_l, t_{l+1}]$.

If $t \in (t_{l+1}, t_{l+2}]$, denote $z(t) = x(t) - y(t) - b_l x(t_l) u_l(t)$ then $z(t)$ satisfies

$$\begin{cases} z'(t) + \sum_{i=1}^{n} p_i(t) z(t - \tau_i) = 0, & t \in (t_{l+1}, t_{l+2}] \\ z(t) = 0, & t \in [t_{l+1} - \tau_n, t_{l+1}] \\ z(t_{l+1}^+) = b_{l+1} x(t_{l+1}), \end{cases}$$

so $z(t) = b_{l+1} x(t_{l+1}) u_{l+1}(t)$ and the formula (4) holds for $t \in (t_{l+1}, t_{l+2}]$.

The rest can be deduced by analogy and the proof of Lemma 1 is complete.

DEFINITION 1. A solution of (1) and (2) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

Remark 1. From Definition 1, a solution $x(t)$ of (1) and (2) is oscillatory if and only if it satisfies at least one of the following conditions:

(a) $x(t)$ has arbitrarily large zeros, that is, for any sufficiently large $T > 0$, there exists $t^* \geq T$, such that $x(t^*) = 0$;

(b) For any sufficiently large integer K, there exists an integer $k \geq K$, such that $x(t_k^+) x(t_k) < 0$.

Remark 2. If for some impulsive points t_k, $x(t_k) = 0$ or $b_k = 0$, then from (2), $x(t_k^+) = x(t_k)$. Thus $x(t)$ is continuous and differentiable at such points t_k.

DEFINITION 2. The zero solution of (1) and (2) is said to be

(a) Stable, if for any $\epsilon > 0$, $\sigma \geq 0$, there exists a $\delta = \delta(\epsilon, \sigma) > 0$ such that $\phi \in PC_\sigma$ and $\|\phi\| < \delta$ imply $|x(t)| < \epsilon$ for $t \geq \sigma$, where $x(t) = x(t, \sigma, \phi)$ is the solution of initial value problem (1), (2), and (3).

(b) Asymptotically stable, if it is stable and for any solution $x(t)$ of (1) and (2), $\lim_{t\to\infty} x(t) = 0$ holds.

3. MAIN RESULTS

Our first result provides sufficient conditions for the nonoscillatory solutions of (1) and (2) to tend to zero as $t \to \infty$.

THEOREM 1. *Assume that*

$$\sum_{k=1}^{\infty} b_k^+ < \infty \tag{9}$$

and there exist positive constants A, B, α, and $r \in [0, \tau_n]$ such that the following conditions are satisfied for sufficiently large t

$$|p_i(t)| \leq A, \qquad i = 1, 2, \ldots, n, \tag{10}$$

$$\sum_{i=1}^{n} p_i(t + \tau_i) \geq B, \tag{11}$$

and

$$\sum_{\tau_i < r} \int_{t-r}^{t-\tau_i} p_i^-(s + \tau_i)\, ds + \sum_{\tau_i > r} \int_{t-\tau_i}^{t-r} p_i^+(s + \tau_i)\, ds \leq \alpha < 1, \tag{12}$$

where $b_k^+ = \max\{b_k, 0\}$, $p_i^+\{s\} = \max\{p_i(s), 0\}$ and $p_i^-(s) = \max\{-p_i(s), 0\}$. Then every nonoscillatory solution of (1) *and* (2) *tends to zero as $t \to \infty$.*

Proof. Choose a positive integer N such that (10), (11), and (12) are satisfied for $t \geq t_N$ and $\sum_{k=N}^{\infty} b_k^+ < 1 - \alpha$. Let $x(t)$ be a nonoscillatory solution of (1) and (2) and, without loss of generality, suppose that $x(t) > 0$ for $t \geq t_N$. Set

$$y(t) = x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t-r} p_i(s+\tau_i) x(s)\, ds - \sum_{t_N \leq t_k < t} b_k^+ x(t_k), \tag{13}$$

then for $t \neq t_k$, $t \neq t_k + \tau_i$, $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots$

$$y'(t) = -\sum_{i=1}^{n} p_i(t - r + \tau_i) x(t-r). \tag{14}$$

Notice that $y(t_k^+) - y(t_k) = (b_k - b_k^+) x(t_k) \leq 0$ for $k = N, N+1, \ldots,$ (14) and (11) imply that $y(t)$ is decreasing on $[t_N, \infty)$. Set $L = \operatorname{lin}_{t\to\infty} y(t)$. We claim that $L \in R$. Otherwise $L = -\infty$, then $x(t)$ must be unbounded by virtue of (10) and (13). Hence it is possible to choose $t^* > t_N + \tau_n$ such that $y(t^*) < 0$ and $x(t^*) = \max\{x(t)\colon t_N \leq t \leq t^*\}$. Then

$$\begin{aligned} 0 &\geq y(t^*) \\ &\geq x(t^*) - \sum_{\tau_i < r} \int_{t^*-r}^{t^*-\tau_i} p_i^-(s+\tau_i) x(s)\, ds \\ &\quad - \sum_{\tau_i > r} \int_{t^*-\tau_i}^{t^*-r} p_i^+(s+\tau_i) x(s)\, ds - \sum_{t_N \leq t_k < t^*} b_k^+ x(t_k) \\ &\geq \left(1 - \alpha - \sum_{k=N}^{\infty} b_k^+\right) x(t^*) > 0 \end{aligned}$$

which is a contradiction. So $L = \lim_{t\to\infty} y(t) \in R$.

We are now ready to prove that $\lim_{t\to\infty} x(t) = 0$. In fact, from (11) and (14) we have

$$y'(t) \leq -Bx(t-r), \qquad t \geq t_N, t \neq t_k, t \neq t_k + \tau_i.$$

Integrating it from t_N to t we have

$$\begin{aligned} B\int_{t_N}^{t} x(s-r)\, ds &\leq -\int_{t_N}^{t} y'(s)\, ds \\ &= y(t_N^+) + \sum_{t_N < t_k < t} \left[y(t_k^+) - y(t_k) \right] - y(t) \\ &\leq y(t_N^+) - L, \end{aligned}$$

which implies that $x \in L^1([t_N, \infty), R)$. Further from (13) and (10) we know $\lim_{t \to \infty} x(t) = L + \sum_{k=N}^{\infty} b_k x(t_k) \in R$ and it must be zero. The proof of Theorem 1 is complete.

In Theorem 1, taking $r = \tau_n$ or $r = 0$ we can obtain the following result.

COROLLARY 1. *Suppose that* (9), (10), (11) *and one of the conditions*

$$\sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i^-(s + \tau_i)\, ds \le \alpha < 1$$

and

$$\sum_{i=1}^{n} \int_{t-\tau_i}^{t} p_i^+(s + \tau_i)\, ds \le \alpha < 1$$

hold. Then every nonoscillatory solution of (1) *and* (2) *tends to zero as* $t \to \infty$.

If the condition (11) is replaced by

$$\sum_{i=1}^{n} p_i(t + \tau_i) \ge 0 \tag{15}$$

in Theorem 1, the conclusion wouldn't hold. For example, consider the equation

$$x'(t) + px(t - \tau_1) - px(t - \tau_2) = 0, \qquad t \ne t_k, \tag{16}$$

$$x(t_k^+) - x(t_k) = b_k x(t_k), \qquad k = 1, 2, \dots, \tag{17}$$

where $p > 0$ is a constant, $b_k \ge 0$, $k = 1, 2, \dots,$ and satisfied (9). It is obvious that (10), (15), and (12) are satisfied, but from the formula (4) using mathematical induction it is easy to prove that the solution of (16) and (17) satisfying the initial value condition

$$x(t) = 1, \qquad t \in [-\tau_2, 0]$$

satisfies $(x)t \ge 1$ for $t \in [0, \infty)$.

If $p_i(t)$, $i = 1, 2, \dots, n$, are nonnegative, the condition (10) and (11) can be weakened.

THEOREM 2. *Assume that* (9) *and*

$$p_i(t) \ge 0, \qquad i = 1, 2, \dots, n \tag{18}$$

and

$$\int^{\infty} \sum_{i=1}^{n} p_i(s + \tau_i)\, ds = \infty \tag{19}$$

hold, then every nonoscillatory solution of (1) *and* (2) *tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and (2), and without loss of generality we assume that $x(t) > 0$ for $t > 0$. First we will prove that $\alpha = \liminf_{t\to\infty} x(t) = 0$. Otherwise $0 < \alpha \le \infty$. Set

$$y(t) = x(t) + \sum_{i=1}^{n-1} \int_{t-\tau_n}^{t-\tau_i} p_i(s+\tau_i) x(s)\, ds - \sum_{0<t_k<t} b_k^+ x(t_k),$$

then for $t \ne t_k$, $t \ne t_k + \tau_i$, $k = 1, 2, \dots,$ $i = 1, 2, \dots, n$,

$$y'(t) = -\sum_{i=1}^{n} p_i(t + \tau_i - \tau_n) x(t - \tau_n).$$

Similar to Theorem 1 we can prove that $\alpha < \infty$ and $L = \lim_{t\to\infty} y(t) \in R$. Then for sufficiently large t we have

$$y'(t) \le -\frac{\alpha}{2} \sum_{i=1}^{n} p_i(t - \tau_n + \tau_i),$$

hence

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i(t - \tau_n + \tau_i) \le \frac{2}{\alpha}[y(t_0) - L],$$

which contradicts (19).

On the other hand, since $x(t)$ is nonincreasing in interval $(t_k, t_{k+1}]$ for sufficiently large k, so

$$\lim_{k\to\infty} x(t_k) = \liminf_{t\to\infty} x(t) = 0.$$

Further

$$\limsup_{t\to\infty} x(t) = \lim_{k\to\infty} x(t_k^+) = \lim_{k\to\infty} (1 + b_k) x(t_k) = 0.$$

The proof of Theorem 2 is complete.

Our next result deals with the asymptotic behavior of oscillatory solutions of (1) and (2) and it's special case improves the corresponding result in [5].

THEOREM 3. *Assume that*

$$\sum_{i=1}^{n} |b_k| < \infty \tag{20}$$

and there exist positive constants Q_1, Q_2 *and* $r \in [0, \tau_n]$ *such that*

$$Q_1 + Q_2 < 1 \tag{21}$$

and for sufficiently large t

$$\sum_{i=1}^{n} p_i(t+\tau_i) \neq 0, \tag{22}$$

$$\sum_{i=1}^{n} \int_{t-\tau_i}^{t} |p_i(s+\tau_i)|\, ds \leq Q_1, \tag{23}$$

and

$$\sum_{i=1}^{n} \int_{t-r}^{t-\tau_i} \operatorname{sgn}(r-\tau_i) |p_i(s+\tau_i)|\, ds \leq Q_2. \tag{24}$$

Then every oscillatory solution of (1) *and* (2) *tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be an oscillatory solution of (1) and (2) and $\mu = \limsup_{t\to\infty}|x(t)|$. First we will prove that $\mu < \infty$. Otherwise $\mu = \infty$. Choose a positive integer N such that (22), (23), and (24) hold for $t \geq t_N$ and

$$\sum_{k=N}^{\infty} |b_k| < \frac{1-Q_1-Q_2}{2}, \tag{25}$$

$$\sup_{t_N+\tau_n \leq s \leq t} |x(s)| = \sup_{t_N \leq s \leq t} |x(s)|, \qquad \textit{for } t \geq t_N + \tau_n.$$

Define function $y(t)$ by

$$y(t) = x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t-r} p_i(s+\tau_i) x(s)\, ds - \sum_{t_N \leq t_k < t} b_k x(t_k),$$

then (14) holds and for $t \geq t_N + \tau_n$

$$\begin{aligned} |y(t)| &\geq |x(t)| - \sum_{i=1}^{n} \int_{t-r}^{t-\tau_i} \operatorname{sgn}(r-\tau_i) |p_i(s+\tau_i) x(s)| ds - \sum_{t_N \leq t_k < t} |b_k x(t_k)| \\ &\geq |x(t)| - \left(Q_2 + \sum_{k=N}^{\infty} |b_k| \right) \sup_{t_N \leq s \leq t} |x(t)|, \end{aligned}$$

which implies

$$\sup_{t_N+\tau_n \leq s \leq t} |y(s)| \geq \left(1 - Q_2 - \sum_{k=N}^{\infty} |b_k| \right) \sup_{t_N \leq s \leq t} |x(s)|. \tag{26}$$

Hence $\limsup_{t\to\infty}|y(t)| = \infty$.

Notice that $x(t)$ is oscillatory and for $k \geq N$, $x(t_k^+)x(t_k) = (1 + b_k)x^2(t_k) \geq 0$, we know that $x(t)$ must have arbitrarily large zeroes. So from (14) we see that $y'(t)$ oscillates and has also arbitrarily large zeroes. Thus there exists $\xi > t_N + 2\tau_n$ such that $|y(\xi)| = \sup_{t_N + \tau_n \leq s \leq \xi}|y(s)|$ and $y'(\xi) = 0$. From (14) and (21) we know $x(\xi - r) = 0$ and integrating (14) from $\xi - r$ to ξ we obtain

$$\begin{aligned} y(\xi) &= y(\xi - r) - \int_{\xi-r}^{\xi} \sum_{i=1}^{n} p_i(s - r + \tau_i)x(s - r)\,ds \\ &= \sum_{i=1}^{n} \int_{\xi-2r}^{\xi-r-\tau_i} p_i(s + \tau_i)x(s)\,ds - \sum_{t_N \leq t_k < \xi - r} b_k x(t_k) \\ &\quad - \sum_{i=1}^{n} \int_{\xi-r}^{\xi} p_i(s - r + \tau_i)x(s - r)\,ds \\ &= -\sum_{i=1}^{n} \int_{\xi-r-\tau_i}^{\xi-r} p_i(s + \tau_i)x(s)\,ds - \sum_{t_N \leq t_k < \xi - r} b_k x(t_k), \end{aligned}$$

which implies

$$|y(\xi)| \leq \left(Q_1 + \sum_{k=N}^{\infty} |b_k|\right) \sup_{t_N \leq s \leq \xi} |x(s)|. \tag{27}$$

From (26) and (27) we have

$$\left(1 - Q_1 - Q_2 - 2\sum_{k=N}^{\infty} |b_k|\right) \sup_{t_N \leq s \leq \xi} |x(s)| \leq 0$$

which contradicts (25). Hence $\mu < \infty$.

Next we will prove that $\mu = 0$. Set

$$z(t) = x(t) + \sum_{i=1}^{n} \int_{t-r}^{t-\tau_i} p_i(s + \tau_i)x(s)\,ds + \sum_{t_k \geq t} b_k x(t_k). \tag{28}$$

Clearly $z(t)$ is bounded and for sufficiently large t

$$|z(t)| \geq |x(t)| - Q_2 \sup_{t-\tau_n \leq s \leq t} |x(s)| - \left|\sum_{t_k \geq t} b_k x(t_k)\right|,$$

thus

$$\lambda = \limsup_{t \to \infty} |z(t)| \geq (1 - Q_2)\mu. \tag{29}$$

On the other hand, we see

$$z'(t) = -\sum_{i=1}^{n} p_i(t-r+\tau_i)x(t-r), \qquad t \neq t_k, t \neq t_k + \tau_i, \quad (30)$$

so $z'(t)$ oscillates and has arbitrarily large zeros. Hence there exists a sequence $\{\xi_m\}$ such that $\lim_{m\to\infty}\xi_m = \infty$, $\lim_{m\to\infty}|z(\xi_m)| = \lambda$, and $z'(\xi_m) = 0$, $m = 1, 2, \dots$. Similar to (27) we can obtain

$$|z(\xi_m)| \leq Q_1 \sup_{\xi_m - 2\tau_n \leq s \leq \xi_m} |x(s)| + \left| \sum_{t_k \geq \xi_m - r} b_k x(t_k) \right|,$$

which implies $\lambda \leq Q_1 \mu$ and, in view of (29),

$$(1 - Q_1 - Q_2)\mu \leq 0.$$

This implies $\mu = 0$ and the proof of Theorem 3 is complete.

In Theorem 3, taking $r = \tau_n$, we can obtain the following result which improves the corresponding result in [5].

COROLLARY 2. *Assume that* (9) *and* (20) *hold and*

$$\limsup_{t\to\infty} \int_{t-\tau_n}^{t} \sum_{i=1}^{n} |p_i(s+\tau_i)|\, ds < 1. \quad (31)$$

Then every oscillatory solution of (1) *and* (2) *tends to zero as* $t \to \infty$.

In the following we will establish the results about the stability of the zero solution of (1) and (2).

Let PC denote the set of functions ψ: $[-\tau_n, 0] \to R$ which has at most finite discontinuity points of the first kind and is continuous from the left at such points. For $\phi \in PC_\sigma$ and $\psi \in PC$, define $\|\phi\| = \sup\{|\phi(s)| : s \in [\sigma - \tau_n, \sigma]\}$ and $\|\psi\| = \sup\{|\psi(s)| : s \in [-\tau_n, 0]\}$ then PC_σ and PC are Banach space. For any $\sigma \geq 0$, $t \geq \sigma$, denote a operator $T(t,\sigma)$: $PC_\sigma \to PC$ by

$$[T(t,\sigma)\phi](s) = x(t+s, \sigma, \phi),$$

where $x(t, \sigma, \phi)$ is the unique solution of initial value problems (1), (2), and (3), then we have the following results.

LEMMA 2. *For any* $\sigma \geq 0$ *and* $t \geq \sigma$ *the operator* $T(t,\sigma)$: $PC_\sigma \to PC$ *is a bounded linear operator.*

Proof. Obviously $T(t,\sigma)$ is a linear operator and we will prove its boundedness.

Notice that the operator denoted by $y(t, \sigma, \cdot)$ is bounded where $y(t, \sigma, \phi)$ is the solution of (5) and (6) (see [6]), and there exists constant $M_0(t, \sigma)$ such that $\|y(t + \cdot, \sigma, \phi)\| \leq M_0(t, \sigma)\|\phi\|$ for any $\phi \in PC_\sigma$.

If $t \in [\sigma, t_l]$ where $l = \min\{k : t_k \geq \sigma\}$, then from (4) we know $x(t + s, \sigma, \phi) = y(t + s, \sigma, \phi)$. Hence

$$\|T(t, \sigma)\phi\| = \|x(t + \cdot, \sigma, \phi)\| = \|y(t + \cdot, \sigma, \phi)\| \leq M_0(t, \sigma)\|\phi\|.$$

If $t \in (t_l, t_{l+1}]$, then

$$x(t + s, \sigma, \phi) = \begin{cases} y(t + s, \sigma, \phi), & \text{if } s \leq t_l - t, \\ y(t + s, \sigma, \phi) + b_l x(t_l) u_l(t + s), & \text{if } s > t_l - t. \end{cases}$$

Hence

$$\begin{aligned} \|T(t, \sigma)\phi\| &\leq \|y(t + \cdot, \sigma, \phi)\| + |b_l| \, \|u_l(t + \cdot)\| \, |x(l)| \\ &\leq M_0(t, \sigma)\|\phi\| + |b_l| \, \|u_l(t + \cdot)\| M_0(t_l, \sigma)\|\phi\| \\ &= M_l(t, \sigma)\|\phi\|, \end{aligned}$$

where $M_l(t, \phi) = M_0(t, \sigma) + |b_l| M_0(t_l, \sigma)\|u_l(t + \cdot)\|$.

Using mathematical induction we can prove that there exists $M_k(t, \sigma)$ such that

$$\|T(t, \sigma)\phi\| \leq M_k(t, \sigma)\|\phi\| \qquad \text{for } t \in (t_k, t_{k+1}]$$

which implies that the operator $T(t, \sigma)$ is bounded and the proof of Lemma 2 is complete.

THEOREM 4. *If all solutions of* (1) *and* (2) *are bounded, then the zero solution of* (1) *and* (2) *is stable*.

Proof. For any $\sigma \geq 0$ and $\phi \in PC_\sigma$, we know from the boundedness of the solutions that there exists a constant $M(\sigma, \phi)$ such that

$$\sup_{t \geq \sigma} |x(t, \sigma, \phi)| \leq M(\sigma, \phi),$$

which implies

$$\sup_{t \geq \sigma} \|T(t, \sigma)\phi\| \leq M(\sigma, \phi).$$

Using the uniform boundedness principle (see [1]) we find that there exists a constant $M^*(\sigma)$ such that

$$\sup_{t \geq \sigma} \|T(t, \sigma)\| \leq M^*(\sigma).$$

Further

$$\sup_{t \geq \sigma} |x(t, \sigma, \phi)| \leq \sup_{t \geq \sigma} \|T(t, \sigma)\phi\|$$
$$\leq \sup_{t \geq \sigma} \|T(t, \sigma)\| \|\phi\| \leq M^*(\sigma)\|\phi\|,$$

which shows that the zero solution of (1) and (2) is stable and the proof of Theorem 4 is complete.

Combining the results of Theorems 1–4 we have the following result.

THEOREM 5. *Assume that conditions* (20)–(24) *and either* (10), (11) *or* (18), (19) *are satisfied. Then the zero solution of* (1) *and* (2) *is asymptotically stable.*

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